GLRT for Cooperative Spectrum Sensing: Threshold Setting in Presence of Uncalibrated Receivers

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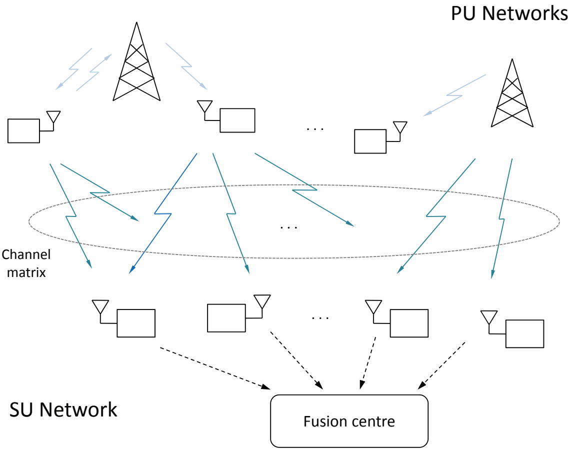
Outline

- Study of GLRT in presence of uncalibrated receivers (SUs with different noise power)
- Statistical description of the test
- Approximated expressions for setting the decision threshold (Neyman-Pearson approach)

Cooperative sensing scenario



System model

Given n_{R} cooperative SUs and n_{T} PUs the output of the receiving antennas at the i-th time instant is

$$\mathbf{y}_i = \mathbf{H}\,\mathbf{x}_i + \mathbf{n}_i$$

where
$\mathbf{n}_i \in \mathbb{C}^{n_{\mathrm{R}}}$ AWGN vector
$\mathbf{x}_i \in \mathbb{C}^{n_{\mathrm{T}}}$ PU transmitted symbol vector; $\mathbf{x}_i \sim \mathcal{CN}(\mathbf{0}, \mathbf{R}_x)$
$\mathbf{H} \in \mathcal{M}_{n_{\mathrm{R}} \times n_{\mathrm{T}}}(\mathbb{C})$ channel gain matrix

Observation matrix from n_{S} snapshots

$$\mathbf{Y} = \left(\mathbf{y}_1 | \cdots | \mathbf{y}_{n_{\mathrm{S}}}\right).$$

GLRT derivation

Under the $\mathcal{H}_j$ hypothesis, with $j = 0, 1$, the likelihood function of $\mathbf{Y}$ is

$$\mathcal{L}(\mathbf{Y}|\boldsymbol{\Sigma}_j) = \frac{1}{\pi^{n_\mathrm{R} n_\mathrm{S}} \left|\boldsymbol{\Sigma}_j\right|^{n_\mathrm{S}}} \exp\left(-n_\mathrm{S} \operatorname{tr}\left\{\boldsymbol{\Sigma}_j^{-1}\mathbf{S}\right\}\right)$$

where the sample covariance matrix (SCM) is defined as $\mathbf{S} = \frac{1}{n_\mathrm{S}}\mathbf{Y}\mathbf{Y}^\mathrm{H}$.
The GLR to detect the hypothesis $\mathcal{H}_0$ is

$$\mathsf{T} = \frac{\left|\widehat{\boldsymbol{\Sigma}}_1\right|}{\left|\widehat{\boldsymbol{\Sigma}}_0\right|} \underset{\mathcal{H}_1}{\overset{\mathcal{H}_0}{\gtrless}} \xi$$

where $0 \leq \xi \leq 1$. $\widehat{\boldsymbol{\Sigma}}_1$ and $\widehat{\boldsymbol{\Sigma}}_0$ are the ML estimates of $\boldsymbol{\Sigma}_0$ and $\boldsymbol{\Sigma}_1$, respectively.

Sphericity Test

Assuming
$\mathcal{H}_0$: $\boldsymbol{\Sigma}_0 = \mathbb{E}\left\{\mathbf{y}_i {\mathbf{y}_i}^{\mathrm{H}} | \mathcal{H}_0\right\} = \sigma^2 \, \mathbf{I}_{n_{\mathrm{R}}}$, i.e. the SU have the same noise power
$\mathcal{H}_1$: No assumptions on $\boldsymbol{\Sigma}_1 = \mathbb{E}\left\{\mathbf{y}_i {\mathbf{y}_i}^{\mathrm{H}} | \mathcal{H}_1\right\}$

The GLRT is

$$\mathsf{T}^{(\mathsf{sph})} = \frac{|\mathbf{S}|}{\left(\mathrm{tr}\{\mathbf{S}\}/n_{\mathrm{R}}\right)^{n_{\mathrm{R}}}} \underset{\mathcal{H}_1}{\overset{\mathcal{H}_0}{\gtrless}} \xi$$

where $\mathbf{S}$ is the sample covariance matrix (SCM) $\mathbf{S} = \frac{1}{n_{\mathrm{S}}} \mathbf{Y}\mathbf{Y}^{\mathrm{H}}$.

Assuming
$\mathcal{H}_0$: $\mathbf{\Sigma}_0$ is diagonal; "unbalanced receivers" case
$\mathcal{H}_1$: No assumptions on $\mathbf{\Sigma}_1$

In this case the GLRT is

$$\mathsf{T}^{(\mathrm{ind})} = \frac{|\mathbf{S}|}{\prod_{k=1}^{n_{\mathrm{R}}} s_{k,k}} \underset{\mathcal{H}_1}{\overset{\mathcal{H}_0}{\gtrless}} \xi$$

where $s_{i,j}$ is the (i, j) element of $\mathbf{S}$.

Detection with uncalibrated receivers

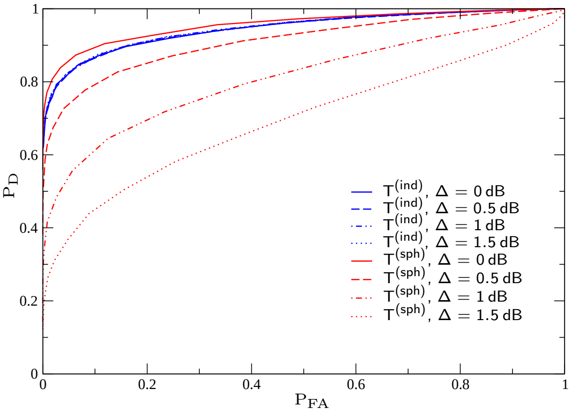


Figure: ROC comparison between $\mathrm{T}^{(\mathrm{ind})}$ and $\mathrm{T}^{(\mathrm{sph})}$ in presence of 4 SUs and a single PU. We assume that the noise power levels in dB at the SU receivers equal $\left(\sigma^2_{\mathrm{ref}}, \sigma^2_{\mathrm{ref}} + \Delta, \sigma^2_{\mathrm{ref}} - \Delta, \sigma^2_{\mathrm{ref}}\right)$. The reference level σ^2_{ref} correspond to $\mathrm{SNR} = -10\,\mathrm{dB}$. $n_{\mathrm{S}} = 500$.

Threshold setting

To design the threshold for both tests we need to know the distribution of $\mathsf{T}^{(\mathrm{ind})}$ and $\mathsf{T}^{(\mathrm{sph})}$ under $\mathcal{H}_0$.

Unfortunately, such statistical distributions are:

- known in closed-form for $\mathsf{T}^{(\mathrm{ind})}$, but expressed as the Meijer G-function multiplied by a normalizing constant[1]
- not known in closed-form for $\mathsf{T}^{(\mathrm{sph})}$, but expressed as an infinite sum[2]

In both cases, such expressions cannot be easily inverted for threshold setting!
Hence, we propose a **moment matching approach**.

[1] M. D. Springer and W. E. Thompson, "The distribution of products of beta, gamma and gaussian random variables," SIAM Journal on Applied Mathematics, vol. 18, no. 4, pp. 712-737, Jun. 1970.

[2] B. N. Nagarsenker and M. M. Das, "Exact distibution of sphericity criterion in the complex case and its percentage points," Communications in statistics, vol. 4, no. 4, pp. 363-374, 1975.

Moments of $T^{(\text{ind})}$

Theorem (On the distribution of the independence test)

Consider the test statistic $T^{(ind)} = |\mathbf{S}| \, / \prod_{k=1}^{n_R} s_{k,k}$, *where* $\mathbf{S} = \{s_{i,j}\}$ *is the SCM of a* n_R*-variate complex Gaussian population with zero mean and diagonal covariance matrix. Then* $T^{(ind)}$ *can be expressed as*

$$T^{(ind)} = T_{n_R} T_{n_R - 1} \cdots T_2 = \prod_{k=2}^{n_R} T_k.$$

where $\{T_k\}_{k=2,\dots,n_R}$ *are independent beta distributed r.v.s with p.d.f.*

$$\begin{cases} \frac{1}{B(n_S - k + 1, k - 1)} \, t^{n_S - k} (1 - t)^{k-2}, & 0 \le t \le 1 \\ 0, & \textit{otherwise.} \end{cases}$$

Moments under $\mathcal{H}_0$

Moments of $\mathrm{T}^{(\mathrm{ind})}$

Based on the previous theorem the moments of $\mathrm{T}^{(\mathrm{ind})}$ can be derived as

$$\mathrm{m}_p^{(\mathrm{ind})} = \prod_{k=2}^{n_\mathrm{R}} \mathbb{E}\left\{\mathrm{T}_k^p\right\}$$

$$\mathrm{m}_p^{(\mathrm{ind})} = \left(\frac{\Gamma(n_\mathrm{S})}{\Gamma(n_\mathrm{S}+p)}\right)^{n_\mathrm{R}-1} \prod_{k=1}^{n_\mathrm{R}-1} \frac{\Gamma(n_\mathrm{S}-k+p)}{\Gamma(n_\mathrm{S}-k)}.$$

Moments of $\mathrm{T}^{(\mathrm{sph})}$ are given by [Nagarsenker and Das, 1975]

$$\mathrm{m}_p^{(\mathrm{sph})} = \frac{n_\mathrm{R}^{n_\mathrm{R} p}\Gamma(n_\mathrm{S} n_\mathrm{R})}{\Gamma(n_\mathrm{S} n_\mathrm{R} + n_\mathrm{R} p)} \prod_{i=1}^{n_\mathrm{R}} \frac{\Gamma(n_\mathrm{S}-i+1+p)}{\Gamma(n_\mathrm{S}-i+1)}.$$

MM-based approximation (1)

We approximate $\mathrm{T}^{(\mathrm{ind})}$ and $\mathrm{T}^{(\mathrm{sph})}$ to beta distributed r.v.s.

Thus the approximated p.d.f. is given by

$$f_{\mathrm{T}}(t) \simeq \begin{cases} \frac{1}{B(\mathrm{a},\mathrm{b})}\, t^{\mathrm{a}-1}\,(1-t)^{\mathrm{b}-1}, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

where $B(\mathrm{a},\mathrm{b}) = \int_0^1 x^{\mathrm{a}-1}(1-x)^{\mathrm{b}-1}\,dx$ is the beta function with parameters a and b:

$$\mathrm{a} = \frac{\mathrm{m}_1\,(\mathrm{m}_2 - \mathrm{m}_1)}{\mathrm{m}_1^2 - \mathrm{m}_2},$$

$$\mathrm{b} = \frac{(1-\mathrm{m}_1)\,(\mathrm{m}_2 - \mathrm{m}_1)}{\mathrm{m}_1^2 - \mathrm{m}_2}.$$

MM-based approximation (2)

Thus $\mathrm{P_{FA}} \triangleq \Pr\{\mathsf{T} < \xi | \mathcal{H}_0\}$ can be expressed as

$$\mathrm{P_{FA}} \simeq \int_0^{\xi} \frac{1}{B(\mathsf{a},\mathsf{b})}\, t^{\mathsf{a}-1}\,(1-t)^{\mathsf{b}-1}\,dt = \widetilde{B}(\mathsf{a},\mathsf{b},\xi)$$

where $\widetilde{B}(\mathsf{a},\mathsf{b},\xi) = \frac{1}{B(\mathsf{a},\mathsf{b})}\int_0^{\xi} x^{\mathsf{a}-1}\,(1-x)^{\mathsf{b}-1}\,dx$ is the regularized beta function.

The decision threshold can be easily calculated as

$$\xi = \widetilde{B}^{-1}\left(\mathsf{a},\mathsf{b},\mathbf{P}_{\mathbf{FA}}^{\mathbf{DES}}\right).$$

Note that $\widetilde{B}^{-1}(\cdot,\cdot,\cdot)$ can be easily computed using standard mathematical software.

MM-based approximation - $T^{(ind)}$

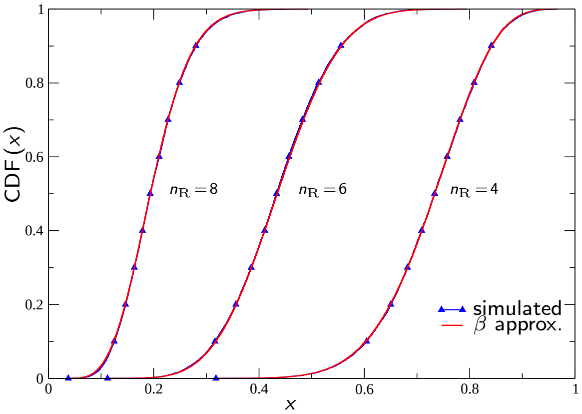


Figure: Comparison between the CDF based on the moment matching strategy and numerically simulated curve for $T^{(ind)}$ under $\mathcal{H}_0$ with $n_S = 20$.

MM-based approximation - $T^{(sph)}$

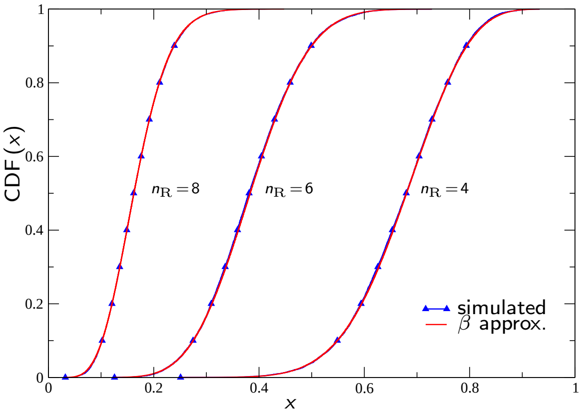


Figure: Comparison between the CDF based on the moment matching strategy and numerically simulated curve for $T^{(sph)}$ under $\mathcal{H}_0$ with $n_S = 20$.

Chi squared approximation comparison

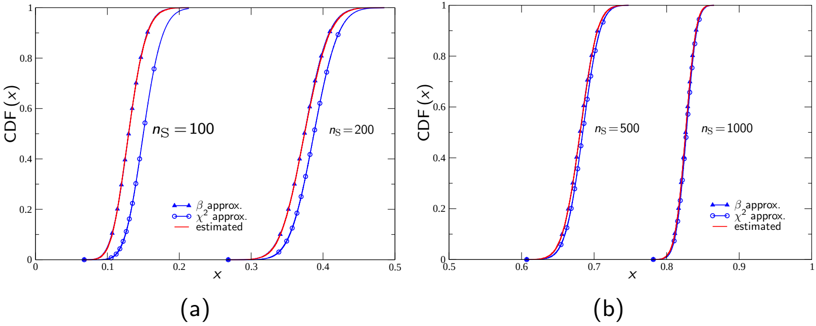


Figure: Comparison among the CDF and the empirically estimated curve for $T^{(ind)}$ under $\mathcal{H}_0$. $n_R = 20$.

Conclusions

We studied the generalized likelihood ratio test (GLRT) for *cooperative spectrum sensing*

- The most proper assumption is that every SU experience a different noise power level (*noise unbalances*)
- $T^{(ind)}$ is robust to noise unbalances $\Rightarrow$ $T^{(ind)}$ should be adopted in place of $T^{(sph)}$
- Both tests can be very well approximated as beta r.v.s. (under null hyp.)
- We provided easy-to-use expressions for setting the decision threshold under the Neyman-Pearson framework.

Thank you!

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A. Mariani, A. Giorgetti, and M. Chiani, "Test of Independence for Cooperative Spectrum Sensing with Uncalibrated Receivers," in *Proc. IEEE Global Commun. Conf.* (GLOBECOM 2012), Anaheim, CA, USA, Dec. 2012.